Dynamical group model of superfluid helium three

1981 J. Phys. A: Math. Gen. 14 2177

(http://iopscience.iop.org/0305-4470/14/9/013)

J. Phys. A: Math. Gen. **14** (1981) 2177–2188. Printed in Great Britain

Dynamical group model of superfluid helium three

A I Solomon
Faculty of Mathematics, The Open University, Milton Keynes, England

Received 9 February 1981

Abstract. We treat a model of an interacting anisotropic superfluid Fermi system, and describe the associated spectrum-generating Lie algebra. This algebra is a direct sum of algebras isomorphic to so(6). Subalgebras correspond to the BCS model of superconductivity (so(3)), and superfluid helium three (so(5)). The spectrum, and so-called unitary states, are expressed in terms of invariants of the algebra.

1. Introduction

The method described in the following pages to treat a model of an anisotropic superfluid Fermi system is based on a similar treatment of an interacting system of bosons given previously by the author (Solomon 1971). The common strategy adopted is as follows. We first write down a model of the interacting system in which we introduce the superfluidity behaviour as a pairing of opposite momentum (but not necessarily opposite spin) operators. We then use a Hartree–Fock approximation to obtain an essentially linearised Hamiltonian. (This linearisation is achieved in the boson case by the Bogoliubov approximation in which the lowest-momentum-state creation and annihilation operators are replaced by a c-number.) The linear, reduced Hamiltonian is then expressed as an element of a Lie algebra g—the spectrum-generating algebra of the model.

It turns out that in both boson and fermion cases g is the direct sum of isomorphic Lie algebras labelled by the momentum suffix k,

$$g = \sum_k g_k,$$

so that the algebraic treatment of the model is essentially governed by the g_k. In the case of interacting bosons, we have

$$g_k \sim \mathrm{so}(2, 1)$$

while in the case of interacting fermions, one obtains

$$g_k \sim \mathrm{so}(6).$$

When the latter case is specialised to spin-singlet pairing ($S = 0$), the BCS model of superconductivity results, for which

$$g_k \sim \mathrm{so}(3),$$

while if one specialises to spin-triplet pairing ($S=1$), a model for superfluid helium three results, for which

$$g_k \sim \mathrm{so}(5).$$

The next step towards diagonalisation is to choose the lowest-dimensional faithful representation of the Lie algebra g_k; the Hamiltonian is then expressed as a matrix M which is 2×2 in the helium four case, and 4×4 in the helium three case. The rotation which effects diagonalisation is an automorphism of the Lie algebra, which we may call the Bogoliubov transformation; however there is no need to perform this rotation explicitly. Instead we may make use of the invariants

$$\mathrm{Tr}\, M^n, \qquad n=1,2,3,\ldots,$$

of which there are only l independent ones associated with a rank-l Lie algebra. For the rank-1 algebra so(2, 1) of helium four (or so(3) of the BCS model), this means that the single invariant $\mathrm{Tr}\, M^2$ leads to the spectrum immediately. In the case of the rank-2 algebra so(5) of helium three, the two invariants, expressed in terms of $\mathrm{Tr}\, M^2$ and $\mathrm{Tr}\, M^4$, give the spectrum in general. The unitary states, which have a degenerate spectrum, correspond to the vanishing of one of the invariants.

As the common strategy in both the boson and fermion cases involves diagonalisation of the Hamiltonian to obtain the energy spectrum by going to a small-dimensional faithful representation of the spectrum-generating algebra, we first exemplify this process by treating the simpler boson case.

2. Superfluid boson model

We take as our model the weakly interacting boson system described by the Hamiltonian $\mathscr{H}=\Sigma_k H_k$, where

$$H_k=\varepsilon_k a_k^+ a_k+\tfrac{1}{2}\sum_{p,q} V_k a_{p+k}^+ a_{q-k}^+ a_p a_q .$$

The operators a_k and a_k^+ represent the annihilation and creation operators for a helium four atom of momentum k; they obey

$$[a_k, a_{k'}^+]=\delta_{kk'}.$$

ε_k is the energy of the atom, and V_k is the Fourier transform of the two-body interaction potential; they satisfy

$$\varepsilon_k=\varepsilon_{-k}, \qquad V_k=V_{-k}.$$

The model is rendered tractable by the assumption of macroscopic occupation of the $k=0$, zero-momentum, state; this enables one to treat the a_0 and a_0^+ operators as the ordinary c-number $\sqrt{N}$, where N is the number density of $k=0$ bosons. This is the assumption which gives rise to the superfluid character of the model. With this simplification the Hamiltonian reduces to $\Sigma_k H_k^{\mathrm{red}}$, where

$$H_k^{\mathrm{red}}=\tfrac{1}{2}(\varepsilon_k+NV_k)(a_k^+a_k+a_{-k}^+a_{-k})+\tfrac{1}{2}NV_k(a_k^+a_{-k}^++a_k a_{-k}).$$

We now exhibit H_k^{red} as an element of a Lie algebra by defining the following operators:

$$X^{(k)} = -\tfrac{1}{2}(a_k^+ a_{-k}^+ + a_k a_{-k}),$$
$$Y^{(k)} = \tfrac{1}{2}\mathrm{i}(a_k^+ a_{-k}^+ - a_k a_{-k}),$$
$$Z^{(k)} = \tfrac{1}{2}(a_k^+ a_k + a_{-k}^+ a_{-k} + 1).$$

These operators obey the commutation rules

$$[X, Y] = -\mathrm{i}Z, \qquad [Y, Z] = \mathrm{i}X, \qquad [Z, X] = \mathrm{i}Y$$

(on suppressing the momentum superscript k), which are the commutation relations of the real Lie algebra su(1, 1) ~ so(2, 1). (The symbol i appears on the right-hand side as a result of the physicist's preference for Hermitian operators.) This then completes the initial part of the strategy outlined in the Introduction, namely to express the Hamiltonian as an element of a Lie algebra g. In this case

$$g \sim \sum_k g_k$$

where each g_k is isomorphic to so(2, 1) or, equivalently, su(1, 1). In terms of the generators of g_k, the reduced Hamiltonian may be written (up to a c-number additive constant) as

$$H_k^{\text{red}} = b_k X^{(k)} + c_k Z^{(k)} \qquad (b_k = -NV_k,\ c_k = NV_k + \varepsilon_k).$$

The form of the energy spectrum may be obtained by a rotation—about the $Y^{(k)}$ direction—and this corresponds precisely to the Bogoliubov transformation (Bogoliubov 1947). However, there is never any need to perform this rotation explicitly, as we mentioned in the Introduction, where we also noted that the final part of the general strategy for obtaining the form of the energy spectrum involves diagonalisation in a low-dimensional faithful representation of the Lie algebra. In the present case we may choose

$$\hat{X} = \frac{1}{2}\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}, \qquad \hat{Y} = -\frac{\mathrm{i}}{2}\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}, \qquad \hat{Z} = \frac{1}{2}\begin{bmatrix} 1 & \\ & -1 \end{bmatrix},$$

as a suitable representation in which, suppressing the k dependence for typographical simplicity, the reduced Hamiltonian is represented by the matrix

$$M = \frac{1}{2}\begin{bmatrix} c & b \\ -b & -c \end{bmatrix}.$$

The single independent invariant in this case is

$$\operatorname{Tr} M^2 = \tfrac{1}{2}(c^2 - b^2)$$

(this would correspond to the Killing form in the adjoint representation). We may therefore diagonalise to $(c^2-b^2)^{1/2}\hat{Z}$ when c^2-b^2 is positive, and to $(b^2-c^2)^{1/2}\hat{X}$ for $c^2-b^2<0$. The former case corresponds to a repulsive potential, and tells us that in the original infinite-dimensional representation the diagonal form of the Hamiltonian is

$$\sum_k (\varepsilon_k^2 + 2NV_k\varepsilon_k)^{1/2} Z^{(k)}.$$

Since the spectrum of $Z^{(k)}$ in the only allowed infinite-dimensional representation

(Solomon 1971) is the natural numbers, this gives the well known discrete excitation spectrum of superfluid helium four. The repulsive case gives the continuous spectrum of $X^{(k)}$.

3. Superfluid fermion model

Recognising that superfluidity in fermion systems arises as a consequence of pair formation in opposite momentum states, we take as our starting point a model Hamiltonian in which only those pairing interactions occur:

$$\mathcal{H}=\sum_{k,\alpha}\varepsilon_k a^+_{k\alpha}a_{k\alpha}+\tfrac{1}{2}\sum_{k,k',\alpha,\beta}V_{kk'}a^+_{k\alpha}a^+_{-k\beta}a_{-k'\beta}a_{k'\alpha}.$$

The fermion annihilation and creation operators $a_{k\alpha}$ and $a^+_{k'\beta}$ obey the anticommutation rules

$$[a_{k\alpha}, a^+_{k'\beta}]_+=\delta_{kk'}\delta_{\alpha\beta}$$

where k, k' are three-momentum labels as before, and the additional suffixes α, β are spin labels which may be either up ($\uparrow$) or down ($\downarrow$). We may reduce this Hamiltonian to exactly solvable form by using the following linearisation procedure; for any two operators A and B we have the identity

$$AB=(A-\langle A\rangle)(B-\langle B\rangle)+A\langle B\rangle+B\langle A\rangle-\langle A\rangle\langle B\rangle$$

where the numbers $\langle A\rangle$, $\langle B\rangle$ are the expectation values in some ground state. To the extent that we may ignore deviations from this ground state, we may approximate

$$AB\sim A\langle B\rangle+B\langle A\rangle$$

(where we have suppressed the additive c-number $\langle A\rangle\langle B\rangle$). Applying this process to our model Hamiltonian leads to the reduced approximate Hamiltonian

$$\mathcal{H}^{\text{red}}=\sum_k H_k \tag{3.1}$$

where

$$H_k=\sum_{\alpha}\tfrac{1}{2}\varepsilon_k(a^+_{k\alpha}a_{k\alpha}+a^+_{-k\alpha}a_{-k\alpha})+\sum_{\alpha,\beta}\tfrac{1}{2}V(k,\alpha,\beta)a^+_{k\alpha}a^+_{-k\beta}+\sum_{\alpha,\beta}\tfrac{1}{2}V^*(k,\alpha,\beta)a_{-k\beta}a_{k\alpha} \tag{3.2}$$

and

$$V(k,\alpha,\beta)=\sum_{k'}\langle V_{k,k'}a_{-k'\beta}a_{k'\alpha}\rangle.$$

Note that the summation in H_k is over spins only, and that the reduced Hamiltonian $\mathcal{H}^{\text{red}}$ has decoupled into a sum of independent (commuting) H_k's—just as in the boson case treated previously. We may therefore treat each H_k individually (suppressing the k subscript for typographical convenience when desirable) and, as a consequence, the spectrum-generating algebra we obtain for $\mathcal{H}^{\text{red}}$ will simply be a direct sum of isomorphic algebras associated with H_k.

In order to identify the spectrum-generating algebra associated with the reduced Hamiltonian $\mathcal{H}^{\text{red}}$, we introduce operators A_i defined by

$$(A_1, A_2, A_3, A_4)=(a_\uparrow, a_\downarrow, a^+_{-\downarrow}, a^+_{-\uparrow})$$

where we have suppressed the three-momentum index k. The operators A_i obey the usual fermion anticommutation relations

$$[A_i, A_j^+]_+ = \delta_{ij} \qquad (i, j = 1, 2, 3, 4).$$

We can now define a set of sixteen operators X_{ij} by

$$X_{ij} = A_i^+ A_j \tag{3.3}$$

which are seen to satisfy the commutation relations

$$[X_{ij}, X_{kl}] = \delta_{jk} X_{il} - \delta_{il} X_{kj}; \tag{3.4}$$

this is a special case of the general result obtained in appendix 1. The commutation relations (3.4) are the defining relations of gl(4, R), the Lie algebra of all real 4×4 matrices, as may be readily seen by choosing the following basis of 16 independent 4×4 matrices e_{ij} with the (r, s) element given by

$$(e_{ij})_{rs} = \delta_{ir}\delta_{js} \qquad (r, s, i, j = 1, 2, 3, 4)$$

so that each matrix possesses precisely the one non-zero entry 1. Clearly the e_{ij} span all real 4×4 matrices, and satisfy

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{kj}.$$

Since the set (3.3) exhausts the operators occurring in the Hamiltonian (3.2), we see that the spectrum-generating algebra associated with H_k for each k is a subalgebra of gl(4, R); more precisely, since from hermiticity only the real combinations U_{ij} can occur, where

$$\begin{aligned} U_{kk} &= X_{kk} && (k = 1, 2, 3, 4) \\ U_{kl} &= X_{kl} + X_{lk} && (k < l \leqslant 4) \\ U_{lk} &= \mathrm{i}(X_{kl} - X_{lk}) && (k < l \leqslant 4), \end{aligned}$$

the required algebra is a subalgebra of u(4), the real algebra of Hermitian 4×4 matrices generated by the U_{ij}.

We now write H_k in terms of the generators X_{ij}:

$$H = \tfrac{1}{2}\varepsilon(X_{11} + X_{22} - X_{33} - X_{44}) + \tfrac{1}{2}(V_{\uparrow\uparrow}X_{14} + V_{\uparrow\downarrow}X_{13} + V_{\downarrow\uparrow}X_{24} + V_{\downarrow\downarrow}X_{23} + \text{Hermitian conjugate})$$

where we have again suppressed the k dependence (as well as an additive constant ε) and written

$$V(k, \alpha, \beta) = V_{\alpha\beta} \qquad \text{for } \alpha, \beta = \uparrow, \downarrow.$$

Following the strategy outlined in the Introduction, we now go to the four-dimensional representation $\hat{X}_{ij} = e_{ij}$, so that $H = \Sigma\, m_{ij} X_{ij}$ is represented by $\hat{H} = \Sigma\, m_{ij} e_{ij}$ where the matrix M of coefficients $(M)_{ij} = m_{ij}$ is given by

$$M = \mathscr{E} + \mathscr{V} \tag{3.5}$$

where

$$\mathscr{E} = \frac{1}{2}\begin{bmatrix} E & 0 \\ 0 & -E \end{bmatrix} \qquad \text{and} \qquad \mathscr{V} = \frac{1}{2}\begin{bmatrix} 0 & V \\ V^+ & 0 \end{bmatrix}.$$

The 2×2 matrices E and V are defined by

$$E=\begin{bmatrix}\varepsilon & 0\\ 0 & \varepsilon\end{bmatrix} \qquad \text{and} \qquad V=\begin{bmatrix}V_{\uparrow\downarrow} & V_{\uparrow\uparrow}\\ V_{\downarrow\downarrow} & V_{\downarrow\uparrow}\end{bmatrix}.$$

Since M is traceless, the spectrum-generating algebra associated with this anisotropic fermion model is a subalgebra of su(4) ~ so(6); we now show that it is indeed so(6). We may write the general 2×2 complex matrix V in terms of Pauli matrices τ_μ, $\mu=0,1,2,3$, as

$$V=\sum_{\mu=0}^{3}(a_\mu+\mathrm{i}b_\mu)\tau_\mu \qquad (a_\mu,\ b_\mu\ \text{real})$$

so that, in terms of the generators $\boldsymbol{T}$, $\boldsymbol{U}$ and $\boldsymbol{E}$ defined in appendix 2, the potential matrix $\mathcal{V}$ is given by

$$\mathcal{V}=\boldsymbol{a}\cdot\boldsymbol{T}-\boldsymbol{b}\cdot\boldsymbol{U}+a_0E_1-b_0E_2$$

with $a_\mu\equiv(a_0,\boldsymbol{a})$, $b_\mu\equiv(b_0,\boldsymbol{b})$. The kinetic energy matrix is

$$\mathscr{E}=\varepsilon E_3.$$

Thus the Hamiltonian matrix M includes the generators $\{E_i, T_i, U_i\}$ of appendix 2; since, for example,

$$[T_i, T_j]=\mathrm{i}e_{ijk}S_k, \qquad [E_1, U_j]=\mathrm{i}W_j,$$

this set closes on the 15 generators $\{\boldsymbol{E},\boldsymbol{S},\boldsymbol{T},\boldsymbol{U},\boldsymbol{W}\}$ of so(6).

Therefore, in the language of the Introduction, the spectrum-generating algebra for this model is g, where

$$g=\sum_k g_k$$

and each $g_k\sim$ so(6).

4. Superfluid helium three model

Both the BCS superconductor model and the superfluid helium three model are obtained from the fermion model of the preceding section by specifying the spin transformation properties of the potential matrix $\mathcal{V}$. Thus, we obtain the BCS model when $\mathcal{V}$ is a spin singlet, and helium three when $\mathcal{V}$ is a spin triplet. It is more convenient to apply the involutive automorphism ϕ of appendix 3 when considering these transformation properties, as under ϕ the spin operator $\hat{\boldsymbol{\sigma}}$ takes the simple form $\boldsymbol{S}$. (This ϕ is *not* the Bogoliubov automorphism.)

Spin-singlet pairing

$$[\hat{\boldsymbol{\sigma}}, \mathcal{V}]=0$$

Applying ϕ, $[\phi(\hat{\sigma}), \phi(\mathcal{V})]=0$, that is $[\boldsymbol{S}, \mathcal{V}']=0$, putting $\mathcal{V}'=\phi(\mathcal{V})$. In general, $\mathcal{V}'=\boldsymbol{a}'\cdot\boldsymbol{T}-\boldsymbol{b}'\cdot\boldsymbol{U}+a_0'E_1-b_0'E_2$ (appendix 3), so in the spin-singlet case $\boldsymbol{a}'=\boldsymbol{b}'=0$, and $\mathscr{V}'$ becomes

$$\mathscr{V}'_s = a'_0 E_1 - b'_0 E_2$$

using the commutation relations of appendix 2.

In this case the Hamiltonian matrix M'_s associated with $\mathscr{V}'_s$ is

$$M'_s = \varepsilon E_3 + a'_0 E_1 - b'_0 E_2.$$

The operators $\{E_1, E_2, E_3\}$ generate the so(3) subalgebra of the BCS model. This algebra has a one-dimensional Cartan subalgebra which we may take to be that generated by E_3. The Bogoliubov transformation in this case is therefore the automorphism sending

$$M'_s \mapsto \mathrm{E} E_3 \tag{4.1}$$

where the coefficient E is here given by an expression similar to that in the boson case of § 1, but now with positive invariant form

$$\mathrm{E} = (\varepsilon^2 + a_0'^2 + b_0'^2)^{1/2}.$$

Since $a_0'^2 + b_0'^2 = a_3^2 + b_3^2 = |V_{\uparrow\downarrow}|^2$, we obtain the well known energy-gap expression

$$\mathrm{E} = (\varepsilon^2 + \Delta^2)^{1/2}, \qquad \Delta = |V_{\uparrow\downarrow}|.$$

The automorphism (3.1) is reflected in the Fock space Hamiltonian (3.1) by the diagonalisation

$$\mathscr{H}^{\mathrm{red}} \mapsto \sum_k (\varepsilon_k^2 + \Delta_k^2)^{1/2} (n_{k\uparrow} + n_{k\downarrow})$$

where $n_{k\alpha} = \overset{+}{a}_{k\alpha} a_{k\alpha}$, $\Delta_k = \Delta_{-k}$.

Spin-triplet pairing

We assume that this is the case for helium three superfluid; $\mathscr{V}$ behaves as a vector under the spin operator $\hat{\boldsymbol{\sigma}}$, or, equivalently, $\mathscr{V}'$ behaves as a vector under $\boldsymbol{S}$. We then have a triplet potential

$$\mathscr{V}'_{\mathrm{T}} = \boldsymbol{a}' \cdot \boldsymbol{T} - \boldsymbol{b}' \cdot \boldsymbol{U} \qquad (a'_0 = b'_0 = 0).$$

The Hamiltonian matrix (3.5) then becomes

$$M'_{\mathrm{T}} = \varepsilon E_3 + \boldsymbol{a}' \cdot \boldsymbol{T} - \boldsymbol{b}' \cdot \boldsymbol{U}.$$

It is shown in appendix 2 that the seven operators $\{E_3, T_i, U_i\}$ close on the so(5) ~ sp(4) algebra generated by $\{S_i, T_i, U_i, E_3\}$; this is therefore the spectrum-generating algebra of the triplet-pairing superfluid helium three model.

It is sometimes convenient to specify the potential $\mathscr{V}'$ by the single complex vector $\boldsymbol{d} = \boldsymbol{a}' + \mathrm{i}\boldsymbol{b}'$; we have

$$d_x = a'_1 + \mathrm{i}b'_1 = -b_2 + \mathrm{i}a_2 = \tfrac{1}{2}(V_{\downarrow\downarrow} - V_{\uparrow\uparrow}),$$

$$d_y = a'_2 + \mathrm{i}b'_2 = b_1 - \mathrm{i}a_1 = -\tfrac{1}{2}\mathrm{i}(V_{\uparrow\uparrow} + V_{\downarrow\downarrow}),$$

$$d_z = a'_3 + \mathrm{i}b'_3 = a_0 + \mathrm{i}b_0 = \tfrac{1}{2}(V_{\uparrow\downarrow} + V_{\downarrow\uparrow}).$$

Without enlarging the so(5) algebra, we may accommodate an external magnetic field term $\boldsymbol{h} \cdot \hat{\boldsymbol{\sigma}}$ in the potential $\mathscr{V}_{\mathrm{T}}$, corresponding to an additional term $\boldsymbol{h} \cdot \boldsymbol{S}$ in $\mathscr{V}'_{T}$.

Similarly, a 'density fluctuation' term

$$\iint \psi^+(x)\rho(x-y)\psi(y)\,\mathrm{d}^3x\,\mathrm{d}^3y$$

in second-quantised field operator form could also be added; but as this corresponds simply to a ρE_3 term in the Hamiltonian, we shall subsume such a term in the energy ε.

We therefore note that the most general superfluid helium three model in the context of the so(5) algebra is given by the Hamiltonian matrix

$$M=\varepsilon E_3+\boldsymbol{a}\cdot\boldsymbol{T}-\boldsymbol{b}\cdot\boldsymbol{U}+\boldsymbol{h}\cdot\boldsymbol{S}$$

in our 4×4 matrix representation, after applying the automorphism ϕ (and dropping the primes and k summation).

5. The spectrum and unitary states

In the previous section we showed that the spectrum-generating algebra of our helium three model is so(5). We can now employ the strategy outlined in the Introduction to obtain the spectrum in terms of the two invariants associated with this rank-2 algebra.

For each momentum $\boldsymbol{k}$, the model Hamiltonian is represented by

$$M=\varepsilon E_3+\boldsymbol{a}\cdot\boldsymbol{T}-\boldsymbol{b}\cdot\boldsymbol{U}+\boldsymbol{h}\cdot\boldsymbol{S} \tag{5.1}$$

(where we have included a magnetic field $\boldsymbol{h}$) which is

$$M=\frac{1}{2}\begin{bmatrix}\varepsilon\tau_0+\boldsymbol{h}\cdot\boldsymbol{\tau} & (\boldsymbol{a}+\mathrm{i}\boldsymbol{b})\cdot\boldsymbol{\tau}\\ (\boldsymbol{a}-\mathrm{i}\boldsymbol{b})\cdot\boldsymbol{\tau} & -\varepsilon\tau_0+\boldsymbol{h}\cdot\boldsymbol{\tau}\end{bmatrix}.$$

We define the following two invariants:

$$I_1=\operatorname{Tr}M^2=\varepsilon^2+a^2+b^2+h^2 \qquad (a^2=\boldsymbol{a}\cdot\boldsymbol{a},\ldots)$$

$$I_2=\operatorname{Tr}M^4-\tfrac{1}{4}I_1^2=(\boldsymbol{a}\times\boldsymbol{b}+\varepsilon\boldsymbol{h})^2+(\boldsymbol{a}\cdot\boldsymbol{h})^2+(\boldsymbol{b}\cdot\boldsymbol{h})^2.$$

By definition the Bogoliubov automorphism sends the Hamiltonian element to a Cartan subalgebra; in this case

$$M\mapsto\lambda E_3+\mu S_3$$

where we have chosen as Cartan subalgebra that generated by $\{E_3, S_3\}$, and λ, μ are real numbers.

Explicitly,

$$M\mapsto\frac{1}{2}\begin{bmatrix}\lambda+\mu & & & \\ & \lambda-\mu & & \\ & & -\lambda+\mu & \\ & & & -\lambda-\mu\end{bmatrix} \tag{5.2}$$

with

$$I_1=\lambda^2+\mu^2, \qquad I_2=\lambda^2\mu^2.$$

The corresponding diagonalisation of the Fock space Hamiltonian $\mathscr{H}^{\mathrm{red}}$ is

$$\mathscr{H}^{\mathrm{red}}\mapsto\sum_k[(I_1+2I_2^{1/2})^{1/2}n_{k\uparrow}+(I_1-2I_2^{1/2})^{1/2}n_{k\downarrow}].$$

The energy spectrum therefore has the form

$$E_k^{\pm} = (\varepsilon_k^2 + \Delta_k^{(\pm)2})^{1/2}$$

where the energy gaps $\Delta_k^{(\pm)}$ are given by

$$\Delta_k^{(\pm)2} = a^2 + b^2 + h^2 \pm 2I_2^{1/2}$$

(all the quantities on the right-hand side being functions of $\boldsymbol{k}$).

The energy spectrum is degenerate with a single energy gap when the invariant I_2 vanishes; this is the case for one of λ or μ vanishing, in which case the square of the matrix M (5.2) is proportional to the unit matrix. These give the so-called unitary states. This occurs (for $\varepsilon \neq 0$) when

$$\boldsymbol{a} \times \boldsymbol{b} + \varepsilon \boldsymbol{h} = 0.$$

In the absence of a magnetic field, an equivalent condition in terms of the complex $\boldsymbol{d}$-vector defined in the previous section is

$$\boldsymbol{d} \times \boldsymbol{d}^* = 0.$$

This is the form of condition given by, for example, Leggett (1975).

More generally, the conjugacy classes of the Hamiltonian matrix (5.1) are parametrised by the real pair (λ, μ), which we may take to satisfy $\lambda \geqslant \mu \geqslant 0$, as the other eigenvalues may be obtained from such a pair by inner automorphisms. Extremal cases are $\mu = 0$ (the unitary states) and $\lambda = \mu$. The latter case (for which one of the excitations has vanishing energy) occurs when the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{h}$ form an orthogonal triad, with $\varepsilon = h$ and $a = b$. In terms of the $\boldsymbol{d}$ vector above, the condition on $\boldsymbol{a}$ and $\boldsymbol{b}$ is $\boldsymbol{d} \cdot \boldsymbol{d} = 0$. In the absence of $\boldsymbol{h}$, this condition leads to a vanishing of one of the two energy gaps. We may rewrite this condition in terms of the potential as

$$V_{\uparrow\uparrow} V_{\downarrow\downarrow} = \tfrac{1}{4}(V_{\uparrow\downarrow} + V_{\downarrow\uparrow})^2.$$

6. Conclusions

We have shown that an anisotropically paired Fermi superfluid can be described by a model Hamiltonian which has an associated dynamical group $\Pi_k\, \mathrm{so}(6)_k$. Imposing spin-zero pairing reduces this to the BCS model with corresponding group $\Pi_k\, \mathrm{so}(3)_k$, while the helium three case, with spin-one pairing, has $\Pi_k\, \mathrm{so}(5)_k$ for the spectrum-generating group. Since, for each k, the helium three spectrum is determined by the rank-2 Lie algebra so(5), this leads to two energy gaps; for unitary states—when one of the two associated algebraic invariants vanishes—we obtain a degenerate one-gap spectrum. The inclusion of additional terms in the model Hamiltonian matrix (5.1), such as a term in the generators W_i of appendix 2 corresponding to a spin-gradient coupling term

$$\int \psi^+(\boldsymbol{x}) \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \psi(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x},$$

would enlarge the spectrum-generating algebra to so(6) and thereby introduce an extra energy gap in general.

It should be noted that this is a zero-temperature model, and so no attempt is made to describe the superfluid transition which is accompanied by a loss of (phase) symmetry; however, just as in the boson case of § 1 where the two physical properties of the system (repulsive potential and attractive potential) are reflected in the two conjugacy classes of the so(2, 1) spectrum-generating algebra ($\hat{Z}$ class and $\hat{X}$ class respectively), one might expect that the various physical states of superfluid helium three would be associated with conjugacy classes in so(5). That this is indeed the case will be shown elsewhere.

Acknowledgment

The author would like to thank Dr John Bolton of the Open University's Physics Department for pointing out that helium three does, in fact, become superfluid and for describing some of its properties.

Appendix 1. Representations in terms of fermion operators

Suppose $\{A_r\}$ is a set of n fermion operators,

$$[A_r, A_s^+]_+ = \delta_{rs} \qquad (r, s = 1, 2, \ldots, n).$$

Let $\{J_\alpha\}$ be an $n \times n$ matrix representation of a Lie algebra g,

$$[J_\alpha, J_\beta] = \sum_\gamma c_{\alpha\beta}^\gamma J_\gamma,$$

with matrix elements $(J_\alpha)_{rs}$ and structure constants $c_{\alpha\beta}^\gamma$. Then a straightforward calculation shows that

$$X_\alpha = \sum_{r,s} A_r^+ (J_\alpha)_{rs} A_s$$

is also a representation of g. Further, if the J_α are Hermitian (use structure constants $\mathrm{i}c_{\alpha\beta}^\gamma$) then so too are the X_α.

We may reproduce the example of § 2 of the text by taking for $\{J_\alpha\}$ the $n \times n$ matrix representation $\{e_{ij}\}$ of gl(n, R),

$$(e_{ij})_{rs} = \delta_{ir}\delta_{js}.$$

Then

$$X_{ij} = \sum_{r,s} A_r^+ (e_{ij})_{rs} A_s,$$

that is

$$X_{ij} = A_i^+ A_j.$$

Appendix 2. Representations of the algebra

From the Pauli spin matrices τ_μ ($\mu = 0, 1, 2, 3$)

$$\tau_0 = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, \qquad \tau_1 = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}, \qquad \tau_2 = \begin{bmatrix} & -\mathrm{i} \\ \mathrm{i} & \end{bmatrix}, \qquad \tau_3 = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix},$$

we may define a 4×4 representation of u(4) by

$$J_{\mu\nu} = \tau_\mu \times \tau_\nu$$

with an analogous representation in terms of fermion operators, following the method of appendix 1. The central element $J_{00} = \tau_0 \times \tau_0$ corresponds to

$$X_{00} = A_1^+ A_1 + A_2^+ A_2 + A_3^+ A_3 + A_4^+ A_4$$

which is (essentially) the total momentum operator. The other 15 elements $\{J_{ij}, J_{i0}, J_{0j}: i, j = 1, 2, 3\}$ generate su(4). It is convenient to separate these 15 generators into 5 triples;

$$\{E_i, S_i, T_i, U_i, W_i\}$$

with

$$E_i = \tfrac{1}{2}\tau_i \times \tau_0, \qquad S_i = \tfrac{1}{2}\tau_0 \times \tau_i, \qquad T_i = \tfrac{1}{2}\tau_1 \times \tau_i,$$

$$U_i = \tfrac{1}{2}\tau_2 \times \tau_i, \qquad W_i = \tfrac{1}{2}\tau_3 \times \tau_i.$$

The S_i may be chosen to play the role of generators of spin (see appendix 3):

$$[S_i, S_j] = \mathrm{i}e_{ijk} S_k, \qquad [S_i, T_j] = \mathrm{i}e_{ijk} T_k, \qquad [S_i, U_j] = \mathrm{i}e_{ijk} U_k, \qquad [S_i, E_j] = 0.$$

The other commutation relations may also readily be obtained. The 15 elements generate the full so(6) (~su(4)) algebra of the anisotropic Fermi superfluid model with Cartan subalgebra $\{E_3, W_3, S_3\}$.

The symplectic algebra sp(4) = u(4) $\cap$ sp(4, C) consists of 4×4 matrices of the form

$$\begin{bmatrix} A & B \\ B^+ & -\tilde{A} \end{bmatrix}$$

where the 2×2 complex matrices obey $A = A^+$, $B = \tilde{B}$ (B transposed). It may be readily verified that the subset

$$\{J_{i\mu}, \mu \neq 2; J_{02}\} = \{\tau_i \times \tau_\mu, \tau_0 \times \tau_2; \mu = 0, 1, 3; i = 1, 2, 3\}$$

has this property. This subset generates a 4×4 representation of the ten-dimensional symplectic subalgebra sp(4) ~ so(5). The generators are clearly isomorphic to

$$\{\tau_\mu \times \tau_i, \tau_2 \times \tau_0; \mu = 0, 1, 3; i = 1, 2, 3\}$$

which may be rotated to the isomorphic set

$$\{\tau_\mu \times \tau_i, \tau_3 \times \tau_0; \mu = 0, 1, 2; i = 1, 2, 3\}.$$

We may rewrite these generators in terms of the previously defined triples

$$\{S_i, T_i, U_i, E_3\}$$

which therefore generate an so(5) subalgebra. This corresponds to the superfluid helium three subalgebra. A maximal Abelian subalgebra (Cartan subalgebra) is $\{E_3, S_3\}$.

Appendix 3.

We may write the spin operator $\boldsymbol{\sigma}$ (for suppressed momentum index k) as

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_+ + \boldsymbol{\sigma}_-$$

with

$$\boldsymbol{\sigma}_+ = \tfrac{1}{2} \sum_{\alpha,\beta} a^+_\alpha \boldsymbol{\tau}_{\alpha\beta} a_\beta, \qquad \boldsymbol{\sigma}_- = \tfrac{1}{2} \sum_{\alpha,\beta} a^+_{-\alpha} \boldsymbol{\tau}_{\alpha\beta} a_{-\beta}$$

(where the + and − suffixes refer to momentum $+k$ and $-k$).

In terms of A_i defined in § 2, we have by explicit evaluation

$$\sigma_1 = \sum_{i,j} A_i^+ (\tfrac{1}{2}\tau_3 \times \tau_1)_{ij} A_j, \qquad \sigma_2 = \sum_{i,j} A_i^+ (\tfrac{1}{2}\tau_3 \times \tau_2)_{ij} A_j, \qquad \sigma_3 = \sum_{i,j} A_i^+ (\tfrac{1}{2}\tau_0 \times \tau_3)_{ij} A_j.$$

Therefore the spin operator is represented in the 4×4 representation of § 2 by

$$\hat{\boldsymbol{\sigma}} = (W_1, W_2, S_3).$$

As this representation is not particularly convenient for calculation, we define an involutive automorphism ϕ by

$$\phi : \mathrm{so}(6) \to \mathrm{so}(6), \qquad \phi^2 = 1,$$
$$g \mapsto RgR^{-1},$$

where

$$R = \exp[\tfrac{1}{2}\mathrm{i}\pi(E_3 + S_3 - W_3)].$$

This transforms the generators of so(6) as follows:

$$\boldsymbol{E} \to (T_3, U_3, E_3), \qquad \boldsymbol{S} \to (W_1, W_2, S_3), \qquad \boldsymbol{T} \to (U_2, -U_1, E_1),$$
$$\boldsymbol{U} \to (-T_2, T_1, E_2), \qquad \boldsymbol{W} \to (S_1, S_2, W_3).$$

Under the automorphism ϕ, the spin operator transforms to $\boldsymbol{S}$,

$$\phi(\hat{\boldsymbol{\sigma}}) = \boldsymbol{S}.$$

The potential matrix $\mathscr{V} = \boldsymbol{a} \cdot \boldsymbol{T} - \boldsymbol{b} \cdot \boldsymbol{U} + a_0 E_1 - b_0 E_2$ becomes

$$\phi(\mathscr{V}) = \boldsymbol{a} \cdot \phi(\boldsymbol{T}) - \boldsymbol{b} \cdot \phi(\boldsymbol{U}) + a_0 \phi(E_1) - b_0 \phi(E_2)$$
$$= \boldsymbol{a}' \cdot \boldsymbol{T} - \boldsymbol{b}' \cdot \boldsymbol{U} + a_0' E_1 - b_0' E_2$$

with

$$\boldsymbol{a}' = (-b_2, b_1, a_0), \qquad \boldsymbol{b}' = (a_2, -a_1, b_0), \qquad a_0' = a_3, \qquad b_0' = b_3.$$

References

Bogoliubov N N 1947 *J. Phys. USSR* **11** 23
Leggett A J 1975 *Rev. Mod. Phys.* **47** 331
Solomon A I 1971 *J. Math. Phys.* **12** 390